

Historical pseudo simplified solution of the Dirac-Coulomb equation

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One of the simplified solutions of the Dirac equations with the pure Coulomb potential given in a paper published in 1985 is pseudo. The original paper solved the Dirac equations by introducing a transformation of functions with two strange parameters a and b to transform the original system of the first-order differential equations into two uncoupled differential equations of second order. However, not only the given eigenvalues sets violate the uniqueness of solution but also the said second-order equations are not any necessarily mathematical deduction. In order to determine the introduced parameters, formally, the author actually introduced some self-contradictory mathematical formulas, such as $\sinh \theta = 2ab$, $\cosh \theta = a^2 + b^2$, $\tanh \theta = -Z\alpha/k$, $a^2 - b^2 = 1$, $b = \sinh(\theta/2)$ and $a = \cosh(\theta/2)$. But one has not known the value of the parameters a and b all the while, whereas the parameters were insensibly deleted in the given second-order Dirac-Coulomb equation last. One cannot recover any result given in the paper by making corresponding correctly mathematical calculations.

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I. INTRODUCTION

It is well known that the exact solutions of the wave equations[1] are very important. Because the judgment on whether or not the approximately solution of the wave equations exist usually depends on the existence of exact solutions and some real laws[2][3] are often obtained from the exact solution of the wave equations. For the Dirac equation for a single particle in the Coulomb field, quantum mechanics textbooks generally adopt the exact solution that was first given by Darwin and Gordon[4][5]. Of course, there are some mathematical problems needing to discuss in the Darwin-Gordon solution[6]. However, some other exact solutions of the second-order Dirac-Coulomb equation appear simplified but are clearly incorrect in character[7][8]. In form, it derived those so-called simplified solution by first transforming the system of the first-order Dirac-Coulomb equation into the Schrödinger-like or the Klein-Gordon-like equations then solving the corresponding second-order differential equations to write the distinguished energy eigenvalues. Nevertheless, one cannot recover the corresponding solution basing on the introduced

mathematical methods in the original papers. It has been pointed out that many given solutions and the corresponding formulas of the energy levels are actually not the necessary mathematical deductions of the said second-order Dirac-Coulomb equation. In addition, it is worse that some so-called second-order Dirac-Coulomb equations given in those published papers can not be yielded from the original system of the Dirac- equations of first-order in the Coulomb field. They are not any necessary mathematical deduction of physics and mathematics yet.

Here we show that a historically formal simplified solution of the Dirac-Coulomb equation given in a paper[9] published in Physical Review 22 years ago is a pseudo solution. For the said second-order Dirac-Coulomb equations in which two equations were written in the same form by using sign “ \pm ”, two eigenvalues set should be given and they are actually different. It is well known that two different sets of the energy eigenvalues for the same quantum system violate the uniqueness of solution. However, in the original paper, only one of the eigenvalues sets was given and the other was thrown out of all reason. This case still exists in other papers today[?][?][12]. In particular, in order to write the second-order Dirac-Coulomb equation, the author introduced two strange parameters a and b and afterward gave some self-contradictory expressions such as $\sinh \theta = 2ab$, $\cosh \theta = a^2 + b^2$, $\tanh \theta = -Z\alpha/k$, $a^2 - b^2 = 1$, $b = \sinh(\theta/2)$ and $a = \cosh(\theta/2)$. However, one has not known the values of parameter a and b . By using the mathematical method introduced therein or by using other correct mathematical methods, one cannot recover any so-called Schrödinger-like or Klein-Gordon-like equations given by the author. Consequently, in the mentioned paper, the claimed simplified solutions of the Dirac-Coulomb equation, the corresponding formula for the energy levels and the corresponding mathematical procedures are pseudo.

II. ORIGINAL FORMAL SIMPLIFIED SOLUTION OF DIRAC-COULOMB EQUATION

Many authors claimed that they obtain simplified solution of the Dirac-Coulomb equation. In 1985, Su considered that, with the use of a simple similarity transformation which brought the radial wave equations of the Dirac-Coulomb problem into a form nearly identical to those of the Schrödinger and Klein-Gordon equations, he derived simplified solutions to the Dirac-Coulomb equation for both the bound and continuum states following the familiar standard procedure adopted in the derivation of the conventional solutions. He considered that to obtain the desired form of the second-order radial equations he could still work with a first-order partial differential equation rather than with the second-order Dirac equation widely employed in the derivation of the simplified solutions, and thus he can avoid the task of reducing the solutions of the second-order

equations to those of the original Dirac equations. Here we only check the mathematical procedure for deducing the energy eigensolutions for the bound state in the original paper. The author first wrote the radial Dirac-Coulomb equation in the following form

$$H'_r \begin{pmatrix} R(r) \\ Q(r) \end{pmatrix} = E \begin{pmatrix} R(r) \\ Q(r) \end{pmatrix} \quad (1)$$

with

$$H'_r = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \quad (2)$$

where

$$\begin{aligned} A' &= mc^2 \cosh \theta + \hbar c \left[\sinh \theta \left(\frac{d}{dr} + \frac{1}{r} \right) - \frac{Z\alpha}{r} \right] \\ B' &= - \left\{ mc^2 \sinh \theta + \hbar c \left[\cosh \theta \left(\frac{d}{dr} + \frac{1}{r} \right) - \frac{k}{r} \right] \right\} \\ C' &= mc^2 \sinh \theta + \hbar c \left[\cosh \theta \left(\frac{d}{dr} + \frac{1}{r} \right) + \frac{k}{r} \right] \\ D' &= - \left\{ mc^2 \cosh \theta + \hbar c \left[\sinh \theta \left(\frac{d}{dr} + \frac{1}{r} \right) + \frac{Z\alpha}{r} \right] \right\} \end{aligned} \quad (3)$$

and $k = \pm (j + \frac{1}{2})$, $\alpha = \frac{e^2}{\hbar c}$ being the fine-structure constant. The strange parameters a and b were claimed the real constants by the author. Then it was introduced that

$$\cosh \theta = \frac{a^2 + b^2}{a^2 - b^2}, \quad \sinh \theta = \frac{2ab}{a^2 - b^2}, \quad a^2 - b^2 > 0 \quad (4)$$

and it was selected that

$$a^2 - b^2 = 1, \quad a = \cosh \left(\frac{\theta}{2} \right), \quad b = \sinh \left(\frac{\theta}{2} \right) \quad (5)$$

with At the same time, the author also introduced the other function

$$\hbar \theta = -Z\alpha/K \quad (6)$$

It was alleged that one attained great simplification in solving the radial equations

$$\begin{aligned} Q(r) &= \left[-\frac{E\tilde{\omega}Z\alpha}{\gamma} + \hbar c \left(\frac{d}{dr} + \frac{1+\tilde{\omega}\gamma}{r} \right) \right] \frac{R(r)}{mc^2 + (j+\frac{1}{2})\frac{E}{\gamma}} \\ R(r) &= \left[\frac{E\tilde{\omega}Z\alpha}{\gamma} + \hbar c \left(\frac{d}{dr} + \frac{1-\tilde{\omega}\gamma}{r} \right) \right] \frac{Q(r)}{mc^2 - (j+\frac{1}{2})\frac{E}{\gamma}} \end{aligned} \quad (7)$$

where $\tilde{\omega} = \mp 1$, $\gamma = \left[(j + \frac{1}{2})^2 - Z^2\alpha^2 \right]^{1/2}$. Finally, without some mathematical calculations, it was also alleged that one should derive the equation

$$\left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \left(\frac{E^2 - m^2 c^4}{\hbar^2 c^2} + \frac{2EZ\alpha}{\hbar c r} \right) - \frac{\gamma^2 \pm \tilde{\omega}\gamma}{r^2} \right] \times \begin{pmatrix} R(r) \\ Q(r) \end{pmatrix} = 0 \quad (8)$$

and the distinguished Dirac formula of the energy levels in the Coulomb field for bound state

$$E/mc^2 = \left[1 + Z^2\alpha^2 / \left(n_r + \sqrt{\left(j + \frac{1}{2} \right)^2 - Z^2\alpha^2} \right)^2 \right]^{-1/2} \quad (9)$$

III. ORIGINAL EIGENSOLUTIONS SETS VIOLATE THE UNIQUENESS OF SOLUTION

In fact, the system of the second-order equations is not always equivalent to the corresponding first-order differential equations. Firstly, we don not know what it means for the parameters a and b and how to eliminate the two parameters to derive the second-order equations (8) from those new definitions such as from (1) to (7). We notice that the equation (8) given in the original article should include four equations as follows

$$\begin{aligned}
\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \left(\frac{E^2 - m^2 c^4}{\hbar^2 c^2} + \frac{2EZ\alpha}{\hbar c r} - \frac{\gamma^2 - \gamma}{r^2} \right) R &= 0 \\
\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\bar{R}}{dr} \right) + \left(\frac{E^2 - m^2 c^4}{\hbar^2 c^2} + \frac{2EZ\alpha}{\hbar c r} - \frac{\gamma^2 + \gamma}{r^2} \right) \bar{R} &= 0 \\
\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dQ}{dr} \right) + \left(\frac{E^2 - m^2 c^4}{\hbar^2 c^2} + \frac{2EZ\alpha}{\hbar c r} - \frac{\gamma^2 - \gamma}{r^2} \right) Q &= 0 \\
\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\bar{Q}}{dr} \right) + \left(\frac{E^2 - m^2 c^4}{\hbar^2 c^2} + \frac{2EZ\alpha}{\hbar c r} - \frac{\gamma^2 + \gamma}{r^2} \right) \bar{Q} &= 0
\end{aligned} \tag{10}$$

Each of the equations has its own eigenvalue and eigensolutions set, and these eigenvalues are usually different from each other. It is incorrect for giving only one of the formulas of the energy levels and alleging to recover the distinguished Dirac formula in mathematical and physical logic. One easily finds the eigenvalue set of the general equations

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Lambda}{dr} \right) + \left(\frac{E^2 - m^2 c^4}{\hbar^2 c^2} + \frac{2EZ\alpha}{\hbar c r} - \frac{\gamma^2 \mp \gamma}{r^2} \right) \Lambda = 0 \tag{11}$$

Because some details are usually ignored by using the corresponding special function, we directly solve every differential equation to find the energy eigenvalues. By introducing the substitution

$$\Lambda = \frac{M}{r} \tag{12}$$

The equation (11) becomes

$$\frac{d^2 M}{dr^2} + \left(-\frac{m^2 c^4 - E^2}{\hbar^2 c^2} + \frac{2Z\alpha}{\hbar c r} E - \frac{\gamma^2 \mp \gamma}{r^2} \right) M = 0 \tag{13}$$

This equation has the asymptotic solutions with $E < mc^2$ satisfying the boundary condition

$$M = \exp \left(-\sqrt{\frac{m^2 c^4 - E^2}{\hbar^2 c^2}} r \right) \quad (r \rightarrow \infty) \tag{14}$$

It is assumed that the formal solution of the equation (13) takes the form

$$M = \exp \left(-\sqrt{\frac{m^2 c^4 - E^2}{\hbar^2 c^2}} r \right) u \tag{15}$$

We have

$$\begin{aligned}\frac{dM}{dr} &= \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right)\left(\frac{du}{dr}-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}u\right) \\ \frac{d^2M}{dr^2} &= \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right)\left(\frac{d^2u}{dr^2}-2\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}\frac{du}{dr}+\frac{m^2c^4-E^2}{\hbar^2c^2}u\right)\end{aligned}\quad (16)$$

Substituting into the equation (13), we obtain

$$\frac{d^2u}{dr^2}-2\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}\frac{du}{dr}+\left(\frac{2EZ\alpha}{\hbar cr}-\frac{\gamma^2\mp\gamma}{r^2}\right)u=0\quad (17)$$

Now, seeking the power series solution, let

$$u=\sum_{n=0}^{\infty}d_nr^{s+n}\quad (18)$$

Hence

$$\frac{du}{dr}=\sum_{n=0}^{\infty}(s+n)d_nr^{s+n-1},\quad \frac{d^2u}{dr^2}=\sum_{n=0}^{\infty}(s+n)(s+n-1)d_nr^{s+n-2}\quad (19)$$

Substitute (18) and (19) into the equations (17), we have

$$\sum_{n=0}^{\infty}\left\{\left[(s+n)(s+n-1)-(\gamma^2\mp\gamma)\right]d_n-\left[2\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}(s+n-1)-\frac{2EZ\alpha}{\hbar c}\right]d_{n-1}\right\}r^{s+n-2}=0\quad (20)$$

It gives the recursive relation of the coefficients of the power series

$$\left[(s+n)(s+n-1)-(\gamma^2\mp\gamma)\right]d_n-\left[2\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}(s+n-1)-\frac{2EZ\alpha}{\hbar c}\right]d_{n-1}=0\quad (21)$$

The power series (18) naturally give the initial value condition: $d_{-1}=d_{-2}=\dots=0$ and $d_0\neq 0$.

Putting let $n=0$ and substituting it into the recursive relation (21) reads $s(s-1)-(\gamma^2\mp\gamma)=0$, it gives

$$s=\frac{1\pm(2\gamma\mp 1)}{2}\quad (22)$$

Thus we have the multi-values of s for the four equations in (10) respectively

$$s_{R_1}=\begin{cases}\gamma\\1-\gamma\end{cases},\quad s_{R_2}=\begin{cases}1+\gamma\\-\gamma\end{cases},\quad s_{Q_1}=\begin{cases}\gamma\\1-\gamma\end{cases},\quad s_{Q_2}=\begin{cases}1+\gamma\\-\gamma\end{cases}\quad (23)$$

Since the wave function has to be normalizable we must choose the value of s to be more than 1 but not only positive sign. In form, for the solutions (23), we can but choose

$$s_{R_1}=\gamma,\quad s_{R_2}=1+\gamma,\quad s_{Q_1}=\gamma,\quad s_{Q_2}=1+\gamma\quad (24)$$

Combining the expression (12), (15), (18), (24), and making using of the value of γ , the completely formal solution of the equations (10) can be written as follows

$$\begin{aligned} R_1 &= \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n-1} \\ R_2 &= \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n} \\ Q_1 &= \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n-1} \\ Q_2 &= \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n} \end{aligned} \quad (25)$$

When $j = 1/2$, $\gamma = \sqrt{1-Z^2\alpha^2} < 1$, the first and the third expression are divergent at the origin of the coordinate system

$$\begin{aligned} \lim_{r \rightarrow 0} R_1 &= \lim_{r \rightarrow 0} \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n-1} = \infty \\ \lim_{r \rightarrow 0} Q_1 &= \lim_{r \rightarrow 0} \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n-1} = \infty \end{aligned} \quad (26)$$

implying that the first equation and the third equation in (10) have no solution which satisfy the boundary conditions. We know that the above divergence have been called “mild divergence” by someone. In (25), the second and the forth expression are finite at the origin of the coordinate system

$$\begin{aligned} \lim_{r \rightarrow 0} R_2 &= \lim_{r \rightarrow 0} \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n} = 0 \\ \lim_{r \rightarrow 0} Q_2 &= \lim_{r \rightarrow 0} \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n} = 0 \end{aligned} \quad (27)$$

implying that the second and the forth equation in (10) seem to have the significative solutions. Comparing (26) and (27) constructs the first kind of contradiction to the second-order differential equation (10).

On the other hand, if we accept the subjective definition of the so-called “mild divergence” or “weak divergence” we would have two eigenvalues set corresponding to the formula of energy levels in the Coulomb field. Form (25), combining the first and the third expression in one form denoted by Λ_1 and combining the second and the forth expression in another form denoted by Λ_2 respectively yield

$$\begin{aligned} \Lambda_1 &= \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n-1} \\ \Lambda_2 &= \exp\left(-\sqrt{\frac{m^2c^4-E^2}{\hbar^2c^2}}r\right) \sum_{n=0}^{\infty} d_n r \sqrt{(j+1/2)^2-Z^2\alpha^2+n} \end{aligned} \quad (28)$$

In order that the wave functions remain normalizable we must require that the series for u so the recursive relation (21) terminate at any term with the power n_r , that means that $d_{n_r} \neq 0$ and

$d_{n_r+1} = d_{n_r+2} = \dots = 0$. Substituting $n = n_{r+1}$ into (21), we have

$$2\sqrt{\frac{m^2c^4 - E^2}{\hbar^2c^2}}(s + n_r) - \frac{2EZ\alpha}{\hbar c} = 0 \quad (29)$$

So that we obtain the Dirac formula in form

$$E = \frac{mc^2}{\sqrt{1 + \frac{Z^2\alpha^2}{(n_r+s)^2}}} \quad (30)$$

According to (24), we finally obtain the eigenvalues of the energy

$$\begin{aligned} E_1 &= \frac{mc^2}{\sqrt{1 + \frac{Z^2\alpha^2}{\left(n_r + \sqrt{(j+1/2)^2 - Z^2\alpha^2}\right)^2}}} \\ E_2 &= \frac{mc^2}{\sqrt{1 + \frac{Z^2\alpha^2}{\left(n_r+1 + \sqrt{(j+1/2)^2 - Z^2\alpha^2}\right)^2}}} \end{aligned} \quad (31)$$

It implies that the solutions of the equations (8) given in the original paper violate the uniqueness of solution of the wave equations. This is the serious mathematical and physical mistakes. In addition, the formula (9) is not a necessarily mathematical and physical deduction. It constructs the second kind of contradiction to the second-order differential equation (10).

IV. THE SECOND-ORDER DIRAC EQUATION (21) IS PSEUDO

In fact, one cannot transform the original radial Dirac-Coulomb equations with first-order into the Schrödinger-like equation or the Klein-Gordon-like equation with the second-order (8). It means that the equation is not equivalent to the original system of first-order Dirac-Coulomb equations. Trying to explain the meaning of the definition such as $\sinh \theta$, $\cosh \theta$ and $\tanh \theta$ make us puzzled, the expressions (4), (5) and (6) read

$$\begin{aligned} \sinh \theta &= 2ab, \quad \cosh \theta = a^2 + b^2, \quad \tanh \theta = -Z\alpha/k \\ a^2 - b^2 &= 1, \quad a = \cosh(\theta/2), \quad b = \sinh(\theta/2) \end{aligned} \quad (32)$$

These expressions given in the original paper early or late are in contradiction with each other. Why did not the author direct give the value of a and b , but deleted them finally expression like magic without any mathematical operation?. Now writing the equations (1) with (2) and (3) in the separate form as follows

$$\begin{aligned} &\{mc^2 \cosh \theta + \hbar c [\sinh \theta (\frac{d}{dr} + \frac{1}{r}) - \frac{Z\alpha}{r}]\} R \\ &- \{mc^2 \sinh \theta + \hbar c [\cosh \theta (\frac{d}{dr} + \frac{1}{r}) - \frac{k}{r}]\} Q = ER \\ &\{mc^2 \sinh \theta + \hbar c [\cosh \theta (\frac{d}{dr} + \frac{1}{r}) + \frac{k}{r}]\} R \\ &- \{mc^2 \cosh \theta + \hbar c [\sinh \theta (\frac{d}{dr} + \frac{1}{r}) + \frac{Z\alpha}{r}]\} Q = EQ \end{aligned} \quad (33)$$

By using the definition (4), it becomes

$$\begin{aligned}
& \frac{2ab\hbar c}{a^2-b^2} \frac{dR}{dr} + \left[\frac{(a^2+b^2)mc^2}{a^2-b^2} + \left(\frac{2ab}{a^2-b^2} - Z\alpha \right) \frac{\hbar c}{r} - E \right] R \\
& - \frac{(a^2+b^2)\hbar c}{a^2-b^2} \frac{dQ}{dr} - \left[\frac{2abmc^2}{a^2-b^2} + \left(\frac{a^2+b^2}{a^2-b^2} - k \right) \frac{\hbar c}{r} \right] Q = 0 \\
& \frac{(a^2+b^2)\hbar c}{a^2-b^2} \frac{dR}{dr} + \left[\frac{2abmc^2}{a^2-b^2} + \left(\frac{a^2+b^2}{a^2-b^2} + k \right) \frac{\hbar c}{r} \right] R \\
& - \frac{2ab\hbar c}{a^2-b^2} \frac{dQ}{dr} - \left[\frac{(a^2+b^2)mc^2}{a^2-b^2} + \left(\frac{2ab}{a^2-b^2} - Z\alpha \right) \frac{\hbar c}{r} - E \right] Q = 0
\end{aligned} \tag{34}$$

Because it is not known for value of the parameter a and b , we cannot obtain the formula of the energy levels in Coulomb field. According to the conflicting expressions (23), by using $a^2 - b^2 = 1$, it can only yield

$$\begin{aligned}
& 2ab\hbar c \frac{dR}{dr} + [(a^2 + b^2) mc^2 - E + (2ab - Z\alpha) \frac{\hbar c}{r}] R \\
& - (a^2 + b^2) \hbar c \frac{dQ}{dr} - [2abmc^2 + (a^2 + b^2 - k) \frac{\hbar c}{r}] Q = 0 \\
& (a^2 + b^2) \hbar c \frac{dR}{dr} + [2abmc^2 + (a^2 + b^2 + k) \frac{\hbar c}{r}] R \\
& - 2ab\hbar c \frac{dQ}{dr} - [(a^2 + b^2) mc^2 + E + (2ab - Z\alpha) \frac{\hbar c}{r}] Q = 0
\end{aligned} \tag{35}$$

This is the original shape of the introduced expression (1). It is clear that this system of differential equations are not equivalent to the original Dirac-Coulomb equations, and one cannot translate them into the so-called Schrödinger-like equations (8). Only when one knows the value of the parameters a and b can translate the system of first-order differential into the corresponding second-order differential equations without the undetermined parameters. Consequently, the second-order equations (8) are the pseudo equations, and the formula of the energy levels (9) is not a necessary mathematical deduction.

V. CONCLUSIONS

In the present paper, we have shown that the original paper published in 1985 used many self-contradictor definition expressions to finally written the so-called simplified solutions of the Dirac-Coulomb equation and all of the given results in the original paper cannot be recovered via the strict mathematical calculating. Such kind of simplified solutions is the pseudo solution of the Dirac equation in the Coulomb field. The corresponding second-order Dirac-Coulomb equations are the pseudo second-order Dirac equation. In fact, it is very simple to solve the system of first-order Dirac-Coulomb equation with the rough boundary condition or the exact boundary condition. We have not understood why many papers treating of the relativistic quantum mechanics seek the corresponding second-order Dirac equation for writing the so-called simplified solutions.

It should be pointed out that the solutions of the original system of first-order Dirac-Coulomb equation are not simplified, and one cannot obtain any simplified solution by constructing the corresponding second-order Dirac-Coulomb equation, unless introducing some incorrect equations. We can use some mathematical theorems of the optimum differential equations[13][14] to discuss such kind of problems[15], and find many papers for constructing second-order Dirac-Coulomb equations are incorrect in mathematical and physical signification. It must be ingeminated that the boundary condition and the uniqueness of solution are very important in solving wave equation. Some classical[16][17][18] and modern[19][20][21] quantum mechanics textbooks have treated these problems[22]. Not anyone should make any mathematical mistakes to spell backward the formula of the energy eigenvalues. The omnifarious mathematical mistakes concealed in the mentioned paper are cross-sectional, implying there are too much similar problems in many published papers[23][24][25].

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